# Forward-Looking Credit Pricing and the Endogenous Cycle in Hours Worked

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October 25, 2026

**Abstract**

U.S. hours worked, unemployment, the investment/output ratio and credit spreads share a recurring cycle of about nine years. We show that whether a model generates this cycle depends on how lenders price household credit. In the model of Beaudry et al. (2020), lenders price loans on current employment. When employment is high, default risk and borrowing costs fall, so households borrow and spend at the same time, and booms and busts recur without persistent outside shocks. We let lenders instead price default risk over the life of the loan, so that borrowing costs depend on expected future employment. The cycle then disappears. The model reproduces it only for loans that last less than about two quarters; for longer loans, the relevant case for household credit, its best fit relies on persistent outside shocks. The benchmark models for policy analysis, Smets and Wouters (2007) and Christiano et al. (2014), have credit premia that do not respond to employment. They produce the cycle only at parameter values that remove shock persistence, and for Smets–Wouters those values are rejected by the likelihood. The search model of Kehoe et al. (2023), in which the price of risk moves independently of employment, produces it in neither unemployment nor risk premia.

**JEL:** E24, E32, E44. **Keywords:** spectral density, limit cycles, strategic complementarity, household credit, rational expectations.

## 1 Introduction

Hours worked, unemployment, the investment/output ratio and credit spreads in the United States share a recurring cycle with a period of about nine years: their spectral densities peak at 32 to 50 quarters (Beaudry et al., 2020). A model can produce such a cycle in two ways. Its shocks can be persistent enough to carry the economy through long swings, or its own propagation can make booms build on themselves and then reverse. Standard estimated models rely on the first, which gives spectra that rise toward low frequencies without a peak (Granger, 1966). This paper asks what the second requires, and finds that the answer depends on how lenders price household credit.

The mechanism of Beaudry et al. (2020) (hereafter BGP) is the clearest case of the second kind. Households borrow to buy durable goods, and a loan to a household member who becomes

*The symbol ⓡ indicates that the order of the authors' names was randomized (Ray ⓡ Robson, 2018). [AEA author-randomization certificate to be added.]

unemployed may not be repaid. Lenders therefore charge a premium that falls when employment is high. When some households spend more, employment rises, borrowing becomes cheaper, and other households spend more as well. The boom continues until the stock of durables is high enough that new purchases fall, employment declines, borrowing becomes expensive, and spending falls together. The cycle recurs with shocks that are nearly serially uncorrelated.

In BGP the loan lasts one period, so the premium depends on current employment only. This matters for two reasons: a boom lowers borrowing costs immediately, and the household's decision places little weight on the future. Lenders on longer loans care about employment over the life of the loan. Late in a boom they expect employment to fall and raise the premium before it does, and late in a downturn they lower it in advance of the recovery. Borrowing costs then respond to where employment is going rather than to where it is, and the feedback from today's spending to today's borrowing costs weakens.

We model this by replacing BGP's one-period loan with a loan that is repaid gradually. A zero-profit lender charges a premium equal to the expected default cost over the life of the loan, which depends on the expected path of employment. One parameter, the average life of the loan, nests BGP when it equals one quarter. We solve the resulting rational-expectations equilibrium, compare it with the data using the same spectral estimator BGP use, and re-estimate all parameters at each loan length.

The cycle disappears as loans lengthen. At BGP's estimates it is gone once loans last slightly more than one quarter. Re-estimating the model recovers it for loans of up to one to one and a half quarters, by raising the inertia that habit in consumption provides; habit cannot provide more, so for longer loans the best-fitting model has no recurring cycle and fits the data through persistent shocks. The equilibrium remains unique at every loan length, so long-term credit does not introduce self-fulfilling fluctuations.

We place this result alongside three models that price credit or risk differently (Table 1). The New Keynesian models of Smets and Wouters (2007) (SW) and Christiano et al. (2014) (CMR) are benchmarks for counterfactual policy analysis. In SW the premium on risky assets is an exogenous shock; in CMR the premium on entrepreneurial loans depends on leverage and on the dispersion of returns. In neither does the price of credit respond to employment. Neither model is built to explain unemployment or risk premia, so for those we use the search model of Kehoe et al. (2023) (KLMP), in which hiring is a long-duration investment and the price of risk depends on a state that moves independently of employment.

None of these models produces the cycle with its estimated or calibrated parameters. SW and CMR produce it in hours only at parameter values that remove the persistence of their shocks. In SW those values imply fluctuations in the investment/output ratio far larger than in the data, and they are rejected by the likelihood of the data the model was estimated on. In CMR one such parameter vector also matches the investment/output ratio, through a single serially uncorrelated investment shock, but places the peak in hours at about twelve rather than nine years. KLMP produces no cycle in unemployment or in the risk premium at its calibration, or at any parameter values we consider that generate realistic unemployment fluctuations. Across the models we study, the recurring cycle appears only when the price of credit responds mainly to current employment.

**Table 1:** Price of credit and employment in each model

| Model | Price of credit or risk |
|---|---|
| SW (2007) | Exogenous AR(1) premium shock; no dependence on employment. |
| CMR (2014) | Premium in a one-period entrepreneurial debt contract; depends on leverage and the cross-sectional risk shock. |
| KLMP (2023) | No credit spread; price of risk is a function of an exogenous habit state; hires valued on expected future productivity. |
| BGP (2020) | Premium on a one-period household loan; expected default cost this period, decreasing in current employment. |
| This paper | Premium on a long-term household loan; expected default cost over the loan's life, a weighted average of expected future employment. |

The paper relates to frequency-domain evaluation and estimation of DSGE models (Christiano and Vigfusson, 2003; Qu and Tkachenko, 2012), to models of endogenous cycles from strategic complementarities (Cooper and John, 1988; Beaudry et al., 2016), to the solution of rational-expectations models with limit cycles (Galizia, 2018), and to the role of household credit in fluctuations (Mian and Sufi, 2018). Our treatment of long-term loans follows the geometric-coupon device of Woodford (2001).

## 2 Data, estimator and loss

**Data.** For the BGP model we use BGP's data: non-farm business hours per capita, $100 \times \log$, 1948Q1–2015Q2 ($T = 270$), and the spread of BAA corporate bond yields over the federal funds rate, in percent, 1954Q3–2015Q2 ($T = 244$). Both are demeaned and not detrended. The SW and CMR results come from companion notes that use FRED data on the same hours concept, $100 \times \log$(HOANBS/CNP16OV), for 1948Q1–2026Q2 ($n = 314$). The two samples differ, so cross-model comparisons of losses are approximate.

**Estimator.** All spectra, data and model, are BGP's estimator: for a demeaned series $x$ of length $T$,

$$I(\omega_j) = \frac{1}{2\pi T}\Big|\sum_{t=1}^{T} x_t e^{-i\omega_j t}\Big|^2, \quad \omega_j = \frac{2\pi j}{1024}, \qquad S(\omega_j) = \sum_{m=-6}^{6} K_m I(\omega_{j+m}), \tag{1}$$

with $K$ a normalized 13-point Hamming kernel. A model spectrum is the expected value of $S$ for a model sample of the data's length: for BGP, the average over simulated samples (as in BGP's estimation); for SW and CMR, the exact expectation computed from the model autocovariances.

**Loss.** We report two losses. $\mathcal{L}$ is BGP's estimation objective: the sum of squared gaps between model and data spectra at periodicities of 2 to 50 quarters, summed over hours and the spread. $\ell$ is the loss used in the SW and CMR notes: the mean squared gap for hours over periodicities 6 to 50 quarters divided by the mean squared level of the data spectrum ($\ell = 0$ is a perfect fit).

## 3 Medium-scale models: hours and the investment/output ratio

SW and CMR are linear rational-expectations models solved by standard methods: every root of the solved system lies inside the unit circle, and the shocks follow AR(1) processes (with news components for CMR's risk shock). The spectrum of a variable $y$ is $s_y(\omega) = \sum_k |H_k(e^{i\omega})|^2 s_{\varepsilon_k}(\omega)$, where $H_k$ is the transfer function from shock $k$ to $y$. With persistent shocks and real or heavily damped complex roots, every term rises toward low frequencies. In SW the premium on risky assets is an exogenous shock; in CMR the external-finance premium is set in a one-period debt contract and depends on entrepreneurial leverage and on the cross-sectional dispersion of returns (the risk shock). In neither model does the price of credit respond directly to employment.

**Hours.** Table 2 summarizes the companion results. At SW's posterior median, the hours spectrum rises monotonically: the solved system's only complex pair has modulus 0.97 and period 376 quarters, and productivity and spending shocks have autocorrelations of 0.97 and 0.95. CMR at its posterior mode also has no peak and too much low-frequency power (64 against 23 in the data at 40 quarters); the risk shock accounts for 53% of hours power at 6–50 quarters. In both models, parameter vectors chosen to match the hours spectrum produce a peak through a complex pair with modulus 0.86–0.98 and remove shock persistence; in SW the price-markup autocorrelation falls from 0.90 to at most 0.21.

**Investment/output.** The parameter vectors that match the hours spectrum do not match the spectrum of the investment/output ratio, which in the data peaks at about 37 quarters (Figure 1). In SW, all three hours-matched vectors generate the hours peak through an investment–capital oscillation; the implied I/Y peak is 13 to 129 times the data's. In CMR, the seed-1 fit generates the hours peak through the risk shock and implies an I/Y spectrum about 4.5 times the data's, peaking at 52 rather than 37 quarters. The seed-0 fit roughly matches I/Y (loss 0.13); it does so with a single serially uncorrelated investment-efficiency shock and the risk shock switched off, and its hours peak is broad and at 47 quarters. Conversely, parameter vectors chosen to match I/Y imply hours peaks of 3.4 to 42,193 in SW and of 257 and 1,723 in CMR, against 23 in the data. Targeting both spectra in SW gives a damped cycle (modulus 0.94) that reproduces neither peak's height.

**Likelihood.** These parameter vectors are inconsistent with the time series the models were estimated on. Relative to SW's posterior median, the Kalman-filter log likelihood of SW's seven observables falls by 23,906 at the hours-matched vector, by 8,962 at the best I/Y-matched vector, and by more than $10^6$ at the jointly matched vector.

## 4 Unemployment and risk premia: the KLMP model

SW and CMR are not designed to account for unemployment or for fluctuations in risk premia. KLMP is. In KLMP, hiring is an investment with long-duration returns because workers accumulate human capital on the job, and the price of risk varies over the cycle through a habit that is exogenous

**Table 2:** Hours and investment/output spectra in SW2007 and CMR (FRED data, 1948Q1–2026Q2)

| | SW2007 | | CMR (2014) | |
|---|---|---|---|---|
| | Posterior median | Hours-matched | Posterior mode | Hours-matched |
| Hours loss $\ell$ (6–50q) | 0.31 | 0.057–0.073 | 8.7 | 0.094–0.097 |
| Hours peak (data: 23, 38q) | none | 20–22, 39–41q | none | 19–20, 47–54q |
| Complex root: modulus | 0.97 | 0.97–0.98 | 0.92 | 0.86–0.93 |
| period (quarters) | 376 | 41–42 | 171 | 49–75 |
| Shock persistence | high | removed | high | removed |
| I/Y loss $\ell$ (6–50q) | 0.93 | 109–9,900 | 10.4 | 0.13; 12.8 |
| I/Y peak (data: 136) | none | 1,714–17,598 | none | 103; 611 |
| Hours peak, I/Y-matched | – | 3.4–42,193 | – | 257; 1,723 |
| Log-likelihood change | – | −23,906 | – | not computed |

Notes: Hours-matched: all estimated parameters chosen to minimize $\ell$ for hours (differential evolution, then a local search), three seeds for SW and two for CMR (CMR values reported as seed 0; seed 1). I/Y in 100 × log units. The log-likelihood change is for SW's seven observables relative to the posterior median; at SW's best I/Y-matched vector it is −8,962. CMR's posterior mode also has a pair with modulus 0.76 and period 44 quarters, too damped to produce a peak.

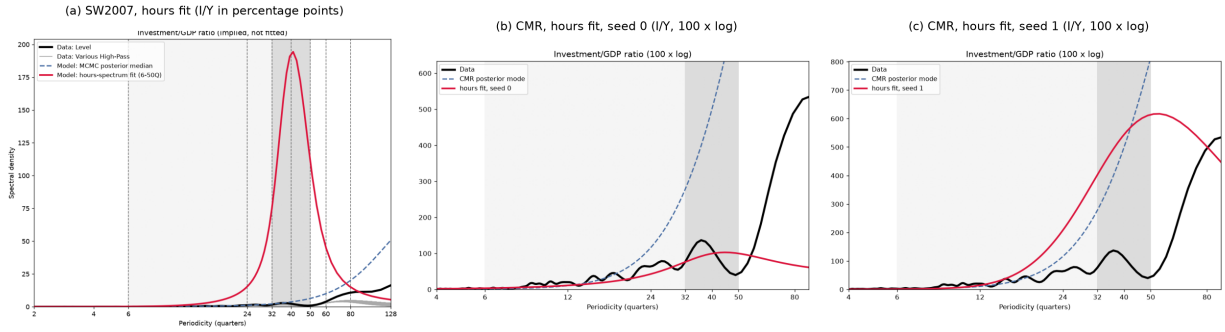



**Figure 1:** Investment/output spectra implied by parameter vectors chosen to match the hours spectrum (red), against the data (black) and each model at its posterior (dashed). Panel (a) is in percentage points of output (data peak about 3); panels (b) and (c) are in 100 × log units (data peak 136). The dark band is 32–50 quarters.

conditional on consumption. KLMP show that the job-finding rate is a weighted average of the prices of claims to future productivity, so that unemployment is volatile when risk premia are.

We solve the baseline model globally, evaluating the pricing kernel at consumption growth equal to productivity growth, which KLMP report is a close approximation. The job-finding rate is then a function of the habit state $s_t$ alone, which follows an AR(1) process with state-dependent volatility. Unemployment follows $u_{t+1} = \sigma(1 - u_t) + (1 - \lambda_{w,t+1})u_t$, as in KLMP, and the risk premium is the price of risk, the conditional standard deviation of the log pricing kernel, $\alpha(1 + \lambda_a(s_t))\sigma_a$. The monthly series are averaged to quarters and passed through the same estimator, on samples of the length of BGP's unemployment (270 quarters) and spread (244 quarters) data. Our solution reproduces the mechanism with about two-thirds of the job-finding volatility KLMP report (standard deviation 4.4 against 6.6 percentage points).

The structure has a direct implication for the spectra. Unemployment is a first-order linear filter,

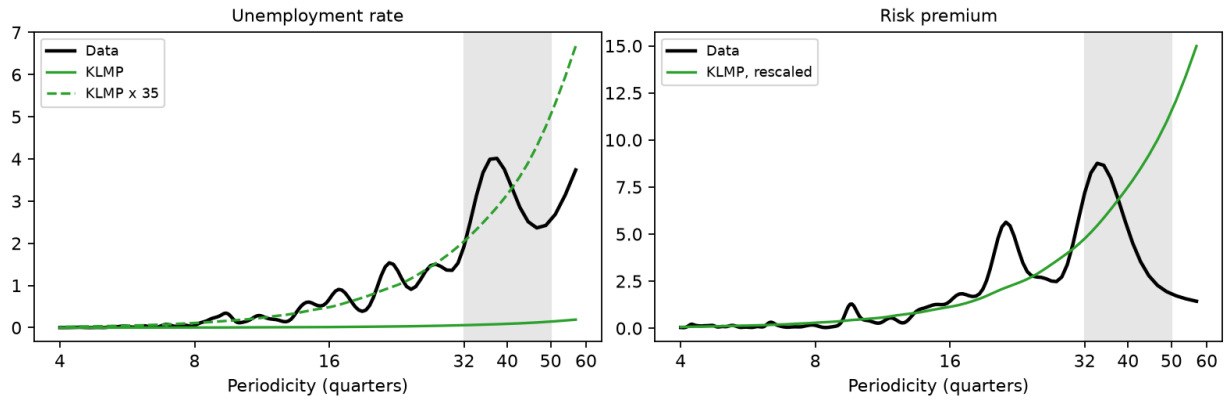



**Figure 2:** KLMP at its calibration: spectral density of the unemployment rate (left) and of the risk premium (right) against BGP's data (unemployment rate; BAA–federal funds spread). Unemployment in percentage points; the dashed line multiplies the model's spectrum by 35, which equates its mean at 6–50 quarters to the data's. The price of risk is rescaled in the same way because its units differ from the spread's. The shaded band is 32–50 quarters.

with a positive root, applied to a static function of an AR(1) state with positive autocorrelation, and the premium is a static function of the same state. If $s_t$ were Gaussian, the spectrum of any function of it would be a positively weighted sum of AR(1) spectra with coefficients $\rho^k$, each decreasing in frequency, so neither series could have an interior peak for any parameter values with $\rho > 0$. The state-dependent volatility makes this approximate, so we check it numerically.

Figure 2 shows the result at KLMP's calibration. Neither spectrum has an interior peak, against peaks at 38 quarters (unemployment) and 34 quarters (spread) in the data. The model's unemployment spectrum is also much smaller than the data's: the standard deviation of quarterly unemployment is 0.58 percentage points against 1.64 in the data, which includes variation in separations that the model holds fixed, and the relative loss is $\ell = 0.95$. We then solve the model on a grid of 324 parameter vectors: annual persistence of the habit state from 0.5 to 0.99, mean surplus-consumption ratio 0.10–0.35, curvature 2–8, human-capital growth on the job of 0–7% and depreciation off the job of 0 or 5% per year. The model has a solution at 214 of them. We define a hump as a local maximum at 24–60 quarters at least 5% above the trough at longer periodicities; in the data the unemployment peak is 1.7 times its trough and the spread peak 6.1 times. No parameter vector produces a hump in the risk premium. Six produce a hump in unemployment; all have curvature $\alpha = 2$ and human-capital growth of 7% a year, and in all six unemployment is nearly constant, with a standard deviation of 0.0004–0.02 percentage points and $\ell > 0.999$. The parameter vector that best fits the unemployment spectrum ($\ell = 0.105$: annual persistence 0.5, $\bar{S} = 0.10$, $\alpha = 8$) generates the data's volatility (standard deviation 1.53 percentage points) and has no hump.

# 5 The BGP model

## 5.1 Equations

BGP's estimated model reduces, by a block-recursive structure, to three equations in log employment $\hat{e}_t$, a scaled durables stock $\hat{X}^\star_t$ and the borrowing premium:

$$\hat{X}^\star_{t+1} = (1-\delta)\hat{X}^\star_t + \psi \hat{e}_t, \tag{2}$$

$$\hat{e}_t = \alpha_1 \hat{X}^\star_t + \alpha_2 \hat{e}_{t-1} + \alpha_3 E_t \hat{e}_{t+1} - \alpha_4 \hat{r}_t + \alpha_4 \mu_t, \tag{3}$$

$$\hat{r}_t = R(\hat{e}_t) \equiv \varrho_1 \hat{e}_t + \varrho_2 \hat{e}_t^2 + \varrho_3 \hat{e}_t^3, \tag{4}$$

with $\mu_t = \rho \mu_{t-1} + \sigma \varepsilon_t$ a discount-factor shock. Equation (3) is the household's consumption Euler equation after substituting the policy rule and the production structure. The coefficients are (Beaudry et al., 2020, Appendix J.1)

$$\alpha_1 = -\frac{\alpha\delta^2(1-\delta-\gamma)}{(1-\delta)\kappa}, \quad \alpha_2 = \frac{\gamma\alpha\delta(1-\delta-\psi)}{(1-\delta)\kappa}, \quad \alpha_3 = \frac{\alpha\delta - \tau\varphi_e}{\kappa}, \quad \alpha_4 = \frac{\tau}{\kappa},$$

$$\varrho_1 = -\frac{(1-\phi+\phi\Phi)e_s}{[e_s+(1-e_s)\phi]^2}, \qquad \Xi = \frac{(1-\phi)e_s}{e_s+(1-e_s)\phi},$$

$$\tau = \frac{(1-\gamma)(\psi+\delta)}{\omega}, \qquad \kappa = (1+\gamma-\psi)\alpha\delta + \tau\Xi,$$

where $\omega$ is the CRRA parameter, $\gamma$ external habit, $\psi$ one minus initial depreciation, $\varphi_e$ the policy response to expected employment, $\phi$ the probability that a defaulting loan is backed by the household, $\Phi$ the recovery cost, $e_s = 0.9417$, $\delta = 0.05$ and $\alpha = 2/3$. Because $\varrho_1 < 0$, a rise in aggregate employment lowers every household's borrowing rate: households' purchases are strategic complements. The "linear RP" model sets $\varrho_2 = \varrho_3 = 0$; the "no friction" model sets $\phi = 1, \Phi = 0$, so $\varrho_1 = 0$; the "canonical" model also sets $\psi = 0$.

BGP show that, in decision rules of the form (3), a Hopf bifurcation, and hence a limit cycle, can occur as the complementarity strengthens only if

$$(1-\delta)\,\alpha_2 > \alpha_3, \tag{5}$$

that is, only if inertia dominates the forward-looking term (Beaudry et al., 2016, Proposition 8). At BGP's estimates $(1-\delta)\alpha_2 = 0.077$ and $\alpha_3 = 0.032$ (nonlinear RP), and 0.116 and 0.049 (linear RP).

## 5.2 Solution and replication

We solve the model as BGP do (Beaudry et al., 2020, Appendix J.2). The state is $s_t = (\hat{X}^\star_t, \hat{e}_{t-1}, \mu_t)$. Following Galizia (2018), the first-order solution is the invariant manifold associated with the three generalized eigenvalues of smallest modulus; this manifold may be locally unstable, in which case the cubic terms of $R$ bound the dynamics on a limit cycle. Second- and third-order terms solve the usual linear perturbation equations, and simulations are not pruned.

**Table 3:** Replication of BGP (2020)

| | Eigenvalues | | Hours peak | | Loss | |
|---|---|---|---|---|---|---|
| Model | This paper | BGP Table 3 | Height, period | BGP Fig. 6 | $\ell$ | $\mathcal{L}$ |
| Nonlinear RP[a] | $1.1032 \pm 0.2164i$ | $1.1032 \pm 0.2164i$ | 18.7, 38q | ≈20, 38q | 0.026 | 267 |
| Linear RP | $0.9269 \pm 0.1368i$ | $0.9258 \pm 0.1372i$ | 18.8, 39q | ≈18, 40q | 0.041 | 398 |
| No friction | 0.5615, 0.9488 | 0.5617, 0.9488 | none | none | 0.070 | – |
| Canonical | 0.5200 | 0.5202 | none | none | 0.071 | – |

Notes: Data peak: 22.9 at 37.9 quarters. [a]With $(\varphi_e, \Phi) = (0.04189, 0.04191)$. Without shocks, the nonlinear RP model converges to a limit cycle with period 39.4 quarters and variance of log hours 7.24 (BGP: about 38 quarters and 7.48); with shocks the variance is 6.83 (BGP: 6.88). Spectra are averages over 3,000 simulated samples.

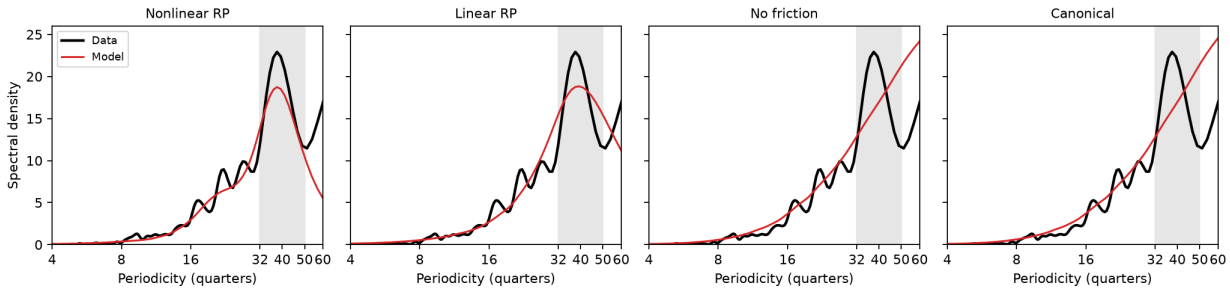



**Figure 3:** Spectral density of hours: data (black) and BGP's four estimated models (red), solved and simulated in this paper. The shaded band is 32–50 quarters.

Table 3 and Figure 3 report the replication. The three linear models reproduce BGP's Table 3 eigenvalues and their Figure 6 spectra at the printed parameter values. The nonlinear model is close to its bifurcation, and its limit cycle is sensitive to the fourth decimal of the estimates. At the printed values its complex pair is $1.0930 \pm 0.2123i$ (reported: $1.1032 \pm 0.2164i$), the limit cycle has a period of 44.5 quarters, and 79% of simulated paths leave the cycle's basin of attraction. Moving $(\varphi_e, \Phi)$ from $(0.0421, 0.0421)$ to $(0.04189, 0.04191)$ reproduces the reported eigenvalues exactly, and with them the reported statistics. We use this parameterization below. No change confined to the printed precision ($\pm 0.00005$ in each structural parameter) reproduces the reported imaginary part, so a small difference from the original code remains.

# 6 Forward-looking pricing of household credit

## 6.1 The premium on a long-term loan

In BGP a household borrows within the period and repays from the period's income. A loan to a member who becomes unemployed defaults with probability $1 - \phi$, and a zero-profit bank charges a premium equal to the expected default cost, $d(e_t)$, which depends on current employment only.

Now let a loan of one unit issued at $t$ amortize geometrically, so that $\kappa^n$ remains outstanding at $t + n$, and let the premium $r_t^L$ be fixed at issuance. In each period the lender loses the expected default cost $d(e_{t+n})$ on the outstanding balance. With discount factor $\beta$, zero expected profit

requires $\sum_{n\geq 0}(\beta\kappa)^n E_t[r_t^L - d(e_{t+n})] = 0$, so

$$r_t^L = (1-\lambda)\sum_{n=0}^{\infty}\lambda^n E_t\, d(e_{t+n}), \qquad \lambda \equiv \beta\kappa, \qquad \hat{r}_t^L = (1-\lambda)R(\hat{e}_t) + \lambda E_t \hat{r}_{t+1}^L, \tag{6}$$

where the second expression applies BGP's approximation of $d$ by $R$ and is written recursively. The average life of the loan is $D = 1/(1-\lambda)$ quarters, and $D = 1$ ($\lambda = 0$) is BGP. The household's borrowing rate in (3) becomes $\hat{r}_t = \hat{r}_t^L$, and its data counterpart is the BAA–federal funds spread, a long-maturity spread.

**Definition 1** (Equilibrium)**.** *Given parameters and $\lambda \in [0,1)$, an equilibrium is a pair of functions $\hat{e}_t = g_e(s_t)$ and $\hat{r}_t^L = g_L(s_t)$ of the state $s_t = (\hat{X}_t^\star, \hat{e}_{t-1}, \mu_t)$ such that, for every $s_t$,*

$$\begin{aligned} g_e(s_t) &= \alpha_1 \hat{X}_t^\star + \alpha_2 \hat{e}_{t-1} + \alpha_3 E_t\, g_e(s_{t+1}) - \alpha_4\, g_L(s_t) + \alpha_4 \mu_t, \\ g_L(s_t) &= (1-\lambda)R\big(g_e(s_t)\big) + \lambda E_t\, g_L(s_{t+1}), \end{aligned}$$

*with $s_{t+1} = \big((1-\delta)\hat{X}_t^\star + \psi g_e(s_t),\, g_e(s_t),\, \rho\mu_t + \sigma\varepsilon_{t+1}\big)$, and the first-order approximation of $(g_e, g_L)$ lies on the invariant manifold of the three generalized eigenvalues of smallest modulus, the remaining two lying outside the unit circle.*

When the three retained eigenvalues lie inside the unit circle, this is the equilibrium of Blanchard and Kahn (1980). When some lie outside, it is Galizia's saddle-cycle solution, which BGP use. We compute $g_e$ and $g_L$ jointly by perturbation, to first order for the linear models and third order for the nonlinear model, and verify that the residuals of both equilibrium conditions are zero at each order. All results below are properties of this equilibrium: its roots, its simulated paths and its spectra.

The household's intertemporal margin is priced at the premium on new long-term borrowing. Outstanding debt and its locked-in rate do not enter as a state, as there is no debt stock in BGP's model either. A model in which the household also carries the stock of long-term debt adds a state variable and is not studied here.

### 6.2 An analytical condition

Expanding (6) to first order in $\lambda$ gives $\hat{r}_t^L = (1-\lambda)R(\hat{e}_t) + \lambda\varrho_1 E_t\hat{e}_{t+1} + O(\lambda^2)$. Substituting into (3),

$$\hat{e}_t = \alpha_1\hat{X}_t^\star + \alpha_2\hat{e}_{t-1} + \underbrace{\big(\alpha_3 + \lambda\alpha_4|\varrho_1|\big)}_{\alpha_3^{\text{eff}}} E_t\hat{e}_{t+1} - (1-\lambda)\alpha_4 R(\hat{e}_t) + \alpha_4\mu_t + O(\lambda^2). \tag{7}$$

The long-term premium leaves the decision rule in BGP's class. It reduces the contemporaneous complementarity by the factor $1-\lambda$ and raises the forward-looking coefficient by $\lambda\alpha_4|\varrho_1|$. Condition (5) becomes

$$(1-\delta)\,\alpha_2 > \alpha_3 + \lambda\,\alpha_4|\varrho_1|. \tag{8}$$

At BGP's estimates $\alpha_4|\varrho_1| = 0.90$ (nonlinear RP) and 0.82 (linear RP), against $\alpha_3 = 0.03$–$0.05$. Condition (8) then fails at $D^* = 1.054$ quarters (nonlinear) and $D^* = 1.089$ quarters (linear).

**Table 4:** Long-term loans at BGP's estimates

| | Nonlinear RP | | | | Linear RP | | |
|---|---|---|---|---|---|---|---|
| $D$ (quarters) | \|root\| | Period | Shocks off | Var(hours) | \|root\| | Period | Var(hours) |
| 1 | 1.124 | 32.4 | limit cycle | 6.69 | 0.937 | 42.9 | 8.18 |
| 1.05 | 0.947 | 15.0 | steady state | 3.81 | 0.899 | 36.3 | 4.68 |
| 1.1 | 0.776 | 17.5 | steady state | 0.74 | 0.833 | 37.4 | 2.27 |
| 1.2 | 0.748 | real | steady state | 0.23 | 0.838 | real | 0.81 |
| 1.5 | 0.878 | real | steady state | 0.06 | 0.901 | real | 0.22 |
| 2 | 0.910 | real | steady state | 0.03 | 0.921 | real | 0.10 |
| 4 | 0.933 | real | steady state | 0.01 | 0.937 | real | 0.04 |
| 20 | 0.945 | real | steady state | 0.01 | 0.946 | real | 0.03 |

Notes: $\alpha_3^{\text{eff}}$ at $D = 1, 1.1, 2$ is 0.032, 0.113, 0.481 (nonlinear) and 0.049, 0.124, 0.461 (linear); $(1-\delta)\alpha_2$ is 0.077 and 0.116. |root|: modulus of the dominant retained eigenvalue of the first-order solution. Period: $2\pi/\arg$ of the complex pair; "real" if both retained roots are real. Shocks off: long-run behavior of the deterministic path. Var(hours): variance of $100\hat{e}_t$ in simulated samples of 270 quarters (the nonlinear model at $D = 1.01$–$1.03$ explodes and is omitted).

**Lemma 1.** *For* $\phi \leq 1$ *and* $0 \leq \psi < 1-\delta$, $\alpha_2 \leq \dfrac{\gamma(1-\delta-\psi)}{(1-\delta)(1+\gamma-\psi)} \leq \dfrac{\gamma}{1+\gamma} < \dfrac{1}{2}$.

*Proof.* $\tau > 0$ and $\Xi \geq 0$, so $\kappa \geq (1+\gamma-\psi)\alpha\delta$, which gives the first inequality. The ratio $(1-\delta-\psi)/(1+\gamma-\psi)$ is decreasing in $\psi$ because $1-\delta < 1+\gamma$, so it is at most $(1-\delta)/(1+\gamma)$. The last inequality holds because $\gamma < 1$. □

Since BGP restrict $\varphi_e < \alpha\delta/\tau$, which makes $\alpha_3 \geq 0$, the lemma and (8) imply that a limit cycle requires

$$\lambda\, \alpha_4 |\varrho_1| < \tfrac{1-\delta}{2} = 0.475 \tag{9}$$

for any parameter values. Long loan lives are compatible with a limit cycle only if the complementarity $\alpha_4|\varrho_1|$ is weak. With BGP's complementarity, (9) limits $D$ to below 2.1 (nonlinear) and 2.4 (linear) quarters even with habit at its bound and $\alpha_3 = 0$. Condition (5) is necessary for a limit cycle, not for complex roots, so (9) does not by itself rule out a damped spectral peak; the numerical results below address that case.

### 6.3 Results at BGP's estimates

Table 4 and Figure 4 hold all other parameters at BGP's estimates and vary $D$. The equilibrium is determinate for every $D$: the smaller eliminated root falls from 1.94 (nonlinear) and 2.69 (linear) at $D = 1$ toward one, and equals 1.0004 at $D = 100$, so long-term loans do not generate sunspot equilibria. The complex pair of the nonlinear model moves inside the unit circle at $D \approx 1.04$, close to $D^*$. Both models lose complex roots at $D \approx 1.17$–$1.18$. The variance of log hours falls from 6.7–8.2 at $D = 1$ to 0.03–0.10 at $D = 2$, and the peak disappears. Between $D = 1.01$ and $D = 1.03$ the nonlinear model's complex pair first moves further out (modulus 1.17) and simulated paths explode.

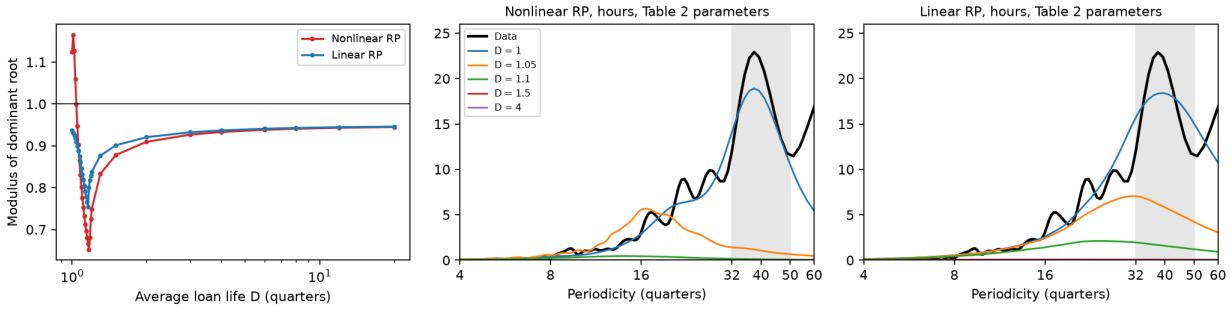



**Figure 4:** Long-term loans at BGP's estimates. Left: modulus of the dominant retained root against average loan life $D$. Center and right: hours spectra for selected $D$ (data in black; the curves for $D = 1.5$ and $D = 4$ are close to zero).

**Table 5:** Linear RP model re-estimated at each loan life

| $D$ | $\mathcal{L}$ | $\ell$ | Roots | Hours peak | $\rho$ | std($\mu$) | $\alpha_4\lvert\varrho_1\rvert$ | $(1-\delta)\alpha_2$ vs. $\alpha_3^{\text{eff}}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 379 | 0.042 | $0.925 \pm 0.138i$ | 18.1, 39q | 0.13 | 0.0002 | 0.83 | $0.087 > 0.078$ |
| 1.5 | 375 | 0.042 | $0.925 \pm 0.138i$ | 18.1, 39q | 0.13 | 0.0010 | 0.64 | $0.314 > 0.253$ |
| 2 | 454 | 0.057 | $0.843 \pm 0.091i$ | 17.2, 49q | 0.84 | 0.0006 | 0.43 | $0.379 < 0.402$ |
| 4 | 497 | 0.064 | 0.760, 0.935 | none | 0.76 | 0.0130 | 0.03 | $0.416 < 0.557$ |
| 8 | 490 | 0.064 | 0.756, 0.932 | none | 0.77 | 0.0195 | 0.03 | $0.424 < 0.554$ |

Notes: $\mathcal{L}$: BGP's objective (hours and spread, 2–50 quarters). $\ell$: relative hours loss, 6–50 quarters. Roots: retained eigenvalues net of the shock root. std($\mu$) $= \sigma/\sqrt{1-\rho^2}$. At BGP's own linear RP estimates $\mathcal{L} = 398$ and $\ell = 0.041$; the no-friction model has $\ell = 0.070$.

## 6.4 Re-estimation

The results above hold the parameters at values estimated for $D = 1$. To ask whether any parameters reproduce the data with long-term loans, we re-estimate the linear RP model at each $D$ by indirect inference (Gourieroux et al., 1993). The estimation follows BGP's objective $\mathcal{L}$: the average of the estimator over 200 simulated samples with common random numbers, a random screen of 300 starting points around BGP's estimates, and Nelder–Mead from the best four. It imposes BGP's parameter restrictions and determinacy.

Table 5 and Figure 5 report the estimates. At $D = 1$ the re-estimated parameters differ from BGP's Table 2 but give the same retained roots and a slightly lower $\mathcal{L}$: the spectra identify the roots more tightly than the structural parameters. At $D = 1.5$ the fit is unchanged. The estimator restores (8) by raising $\gamma$ to 0.89, which raises $(1-\delta)\alpha_2$ to 0.31, and by lowering the complementarity to 0.64. At $D = 2$ habit cannot supply enough inertia ($0.379 < 0.402$). The roots remain complex but more damped, the hours peak moves to 49 quarters, and the shock autocorrelation rises from 0.13 to 0.84. At $D = 4$ and $D = 8$ the estimator sets the complementarity to about 0.03, the retained roots are real, the hours spectrum has no interior peak, and the loss $\ell = 0.064$ is close to the no-friction model's 0.070. The fit then relies on shocks with autocorrelation 0.76–0.77 and an unconditional standard deviation 70 to 100 times that at $D = 1$, the configuration of the medium-scale models in Section 3.

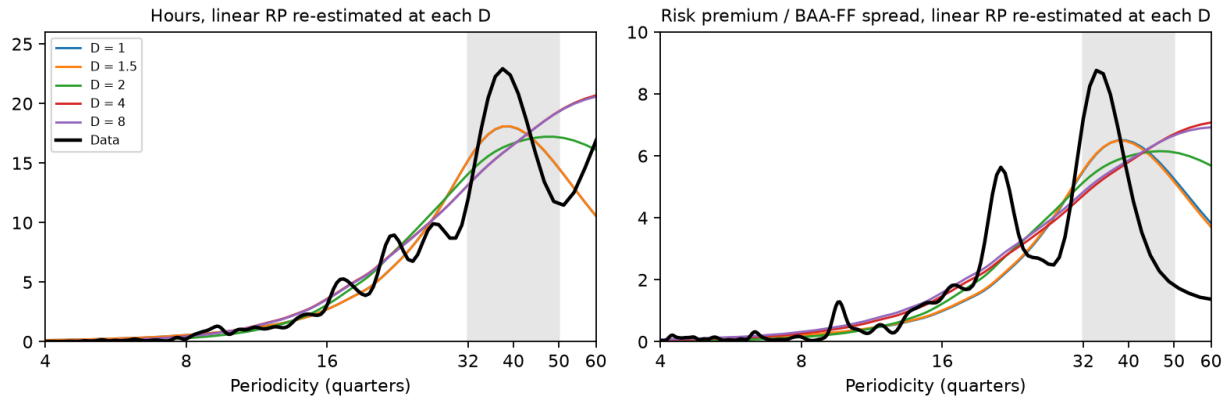



**Figure 5:** Linear RP model re-estimated at each average loan life $D$: hours (left) and the premium against the BAA–federal funds spread (right). Data in black. The $D = 1$ and $D = 1.5$ curves coincide.

**Table 6:** Nonlinear RP model re-estimated at each loan life

| $D$ | $\mathcal{L}$ | Retained roots | \|root\| | Shocks off | $\rho$ | $\alpha_4\|\varrho_1\|$ | $(1-\delta)\alpha_2$ vs. $\alpha_3^{\text{eff}}$ |
|---|---|---|---|---|---|---|---|
| 1 | 223 | $1.110 \pm 0.211i$ | 1.130 | cycle, 39q | 0.00 | 0.89 | $0.086 > 0.034$ |
| 1.05[a] | 454 | $0.879 \pm 0.105i$ | 0.885 | steady state | 0.83 | 0.52 | $0.221 < 0.278$ |
| 1.1 | 225 | $1.002 \pm 0.260i$ | 1.035 | cycle, 31q | −0.02 | 0.83 | $0.145 > 0.102$ |
| 1.5 | 470 | 0.760, 0.838 | 0.838 | steady state | 0.88 | 0.01 | $0.386 < 0.603$ |
| 2 | 428 | 0.822, 0.912 | 0.912 | steady state | 0.66 | 0.07 | $0.417 < 0.530$ |
| 4 | 458 | 0.694, 0.930 | 0.930 | steady state | 0.89 | 0.05 | $0.396 < 0.573$ |
| 8[b] | 559 | 0.487, 0.950 | 0.950 | steady state | 0.85 | 0.00 | – |

Notes: Same procedure as Table 5, with ten parameters. Shocks off: long-run behavior of the deterministic path. [a]Local optimum: the $D = 1.1$ run attains $\mathcal{L} = 225$. [b]Degenerate estimate (zero complementarity, diverging shock variance).

**Nonlinear model.** Table 6 repeats the exercise for the nonlinear RP model, estimating $\varrho_2$ and $\varrho_3$ as well and requiring simulated paths to remain bounded; the objective is the same spectral distance (BGP add five moments for this model, which we omit). At $D = 1$ the estimate has a limit cycle with a period of 39 quarters and $\mathcal{L} = 223$. At $D = 1.1$ the estimator finds a limit cycle with a shorter period (31 quarters) and the same fit. The $D = 1.05$ run stops at a local optimum with a worse fit. From $D = 1.5$ onward the estimator sets the complementarity close to zero, the retained roots are real, the model converges to its steady state without shocks, and $\mathcal{L}$ rises to 428–559, with shock autocorrelations of 0.66–0.89. At $D = 8$ the estimate is degenerate: the complementarity is zero and the shock variance diverges.

## 7 Conclusion

At their estimated posteriors, SW and CMR do not produce the spectral peak in hours. They reproduce it only at parameter values that remove shock persistence and that the time-series likelihood rejects, and in SW those values imply an investment/output spectrum one to two orders of magnitude larger than the data's. BGP's model produces the peak with nearly serially uncorre-

lated shocks. It does so because the premium that carries the complementarity responds to current employment and because the forward-looking term in the household's decision rule is small. When lenders price default over the life of the loan, the complementarity enters through expected future employment. The model's forward-looking coefficient then rises by $\lambda\alpha_4|\varrho_1|$, and the condition for endogenous cycles fails at loan lives slightly above one quarter at BGP's estimates. Re-estimation extends the range to 1.1 (nonlinear) and 1.5 (linear) quarters; the bound $\alpha_2 < \gamma/(1+\gamma)$ prevents habit from extending it further. For longer loans the best-fitting version of the model has no endogenous cycle and fits the hours spectrum through persistent shocks, as SW and CMR do. KLMP, in which hiring is valued on expected future productivity and the price of risk depends on an exogenous state, produces neither the unemployment nor the risk-premium peak at any parameter vector we consider that generates material unemployment volatility.

Three limitations apply. The household's borrowing rate is the premium on new long-term loans; a model with a stock of long-term debt would add a state variable and could add a second accumulation channel. The nonlinear model is re-estimated on the spectra only, without the five additional moments BGP use. The SW and CMR results use a longer sample than the BGP and KLMP results.

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# A Computation

The code is in `bgp/model.py` (model, perturbation solver, simulation) and `bgp/spectra.py` (data, estimator); the results are produced by `scripts/bgp_replicate.py`, `scripts/bgp_long_term_premium.py` and `scripts/bgp_paper_figures.py`. The solver writes the policy functions as polynomials of degree three in $(\hat{X}^\star, \hat{e}_{-1}, \mu/\sigma)$ and a perturbation scale on the shock. Expressing the shock state in units of its innovation standard deviation keeps the third-order system well conditioned. Generalized eigenvalues come from the QZ decomposition of the linearized system; the retained set is the conjugation-closed set of three eigenvalues with the smallest maximum modulus. A first-order cross-check against the scalar characteristic polynomial of the linearized model with (6) agrees to four decimals. Tests are in `tests/test_bgp.py`.